Measure Theory with Ergodic Horizons
Lecture 9

Theorem. Every finite Borel measure μ on a metric space X is strongly regular.

Proof. Let $\mathcal{S}$ be the collection of measurable sets $B \subseteq X$ that satisfy:

$$\mu(B) = \inf\{\mu(U) : U \supseteq B \text{ open}\} \quad \text{(outer regular)}$$
$$= \sup\{\mu(C) : C \subseteq B \text{ closed}\} \quad \text{(inner regular)}$$

Claim (a). $\mathcal{S}$ contains every open set.

Proof. The outer regularity is trivial and inner regularity follows from the fact that in metric spaces open sets are F_σ. Indeed write an open set U as $\bigcup_n C_n$ where each C_n is closed; by replacing C_n with $\bigcup_{i \le n} C_i$, we may assume that the C_n are increasing and hence $\mu(U) = \lim_{n\to\infty} \mu(C_n)$ by monotone convergence. $\square$

Claim (b). $\mathcal{S}$ is closed under complements.

Proof. Follows from the fact that the complements of open/closed sets are closed/open. $\square$

Claim (c) $\mathcal{S}$ is closed under finite unions, hence is an algebra.

Proof. Open/closed sets are closed under finite unions. $\square$

Claim (d). $\mathcal{S}$ is closed under ctbl unions, hence is a σ-algebra.

Proof. Let A_n be sets in $\mathcal{S}$. By claim (c), we may replace each A_n with $\bigcup_{i \le n} A_i$ and assume that the A_n are increasing.

For outer regularity, let $U_n \supseteq A_n$ be open sets with $\mu(U_n \setminus A_n) < \varepsilon/2^{n+1}$, so $\mu(\bigcup_n U_n \setminus \bigcup_n A_n) \le \sum_n \mu(U_n \setminus A_n) < \varepsilon$ and $\bigcup_n U_n \supseteq \bigcup_n A_n$ is open.

For inner regularity, let $C_n \subseteq A_n$ be closed sets with $\mu(A_n \setminus C_n) < 1/n$. Then for large enough $k \in \mathbb{N}$, $\mu(A_k) \approx_{\varepsilon/2} \mu(\bigcup_n A_n)$ by monotone convergence, and $\mu(C_k) \approx_{\varepsilon/2} \mu(A_k)$, so $C_k \subseteq \bigcup_n A_n$ is closed and $\mu(\bigcup_n A_n \setminus C_k) < \varepsilon$. □

Thus, $\mathcal{S}$ contains all Borel sets. Now if M is a μ-measurable set, then there are Borel sets $B_-, B_+ =_\mu M$ and $B_- \subseteq M \subseteq B_+$. Then take open $U \supseteq B_+$ and closed $C \subseteq B_-$ with $\mu(C) \approx_\varepsilon \mu(B_-) = \mu(M) = \mu(B_+) \approx_\varepsilon \mu(U)$. These U and C witness the regularity of M. Thus $\mathcal{S} = \mathrm{Meas}_\mu$. □

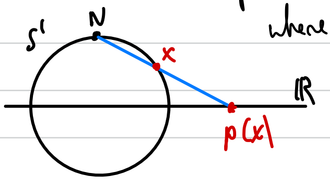
Def. Say that a Borel measure μ on a metric space X is **σ-finite by open sets** if X is a ctbl union of open sets of finite measure.

Cor. If a Borel measure μ on a metric space X is σ-finite by open sets, then μ is strongly regular.

Proof. Let $X = \bigcup_n U_n$ where each U_n is open and of finite measure. Let A be and measurable set. Then for each $n \in \mathbb{N}$, there is an open set $V_n \subseteq U_n$ relative to U_n such that $V_n \supseteq A \cap U_n$ and $\mu(V_n \setminus A) < \varepsilon/2^{n+1}$. But because U_n is open in X, V_n too is open in X. Thus, $\bigcup_n V_n \supseteq A$ is open and $\mu(\bigcup_n V_n \setminus A) \le \sum_n \mu(V_n \setminus A) < \varepsilon$. Thus strong outer regularity holds.
For inner regularity, let $U \supseteq A^c$ be open such that $\mu(U \setminus A^c) < \varepsilon$. But then $U^c \subseteq A$ is closed and $\mu(A \setminus U^c) = \mu(A \cap U) = \mu(U \setminus A^c)$. □

Caution. It's not true that all σ-finite Borel measures are regular, as the following example shows.

Example. Let $X :=$ the one-point compactification of $\mathbb{R}$, i.e. $X = \mathbb{R} \cup \{\infty\}$ with the metric copied from S^1 via the stereographic projection $p: S^1 \to \mathbb{R} \cup \{\infty\}$, where $S^1 \setminus \{N\} \xrightarrow{\sim} \mathbb{R}$ and $p(N) = \infty$. The open sets in X are all open sets in $\mathbb{R}$ together with the sets of the form $\{\infty\} \cup U$ where $U \subseteq \mathbb{R}$ is open in $\mathbb{R}$ and



contains a set of the form $(-\infty, a) \cup (b, \infty)$ for some $a, b \in \mathbb{R}$. Let μ be the Borel measure on X defined by $\mu|_{\mathbb{R}} := \lambda$ (Lebesgue) and $\mu(\{\infty\}) := 0$. Since λ on $\mathbb{R}$ is σ-finite, μ on X is also σ-finite, indeed

$$X = \{\infty\} \cup \bigcup_{n \in \mathbb{Z}} [n, n+1)$$

and each set in this union has finite measure. Yet, μ is outer regular: $\mu(\{\infty\}) = 0$ but $\mu(U) = \infty$ for any open set $U \ni \infty$ because then U contains a set of the form $(-\infty, a) \cup (b, \infty)$.
Observe also that X is not σ-finite by open sets because if $X = \bigcup_n U_n$ where U_n is open, then $\infty \in U_n$ for some $n \in \mathbb{N}$ hence $\mu(U_n) = \infty$.

Def. Let X be a topological space and let μ be a Borel measure on it. We say that:

- μ is *finite on compact sets* if $\mu(K) < \infty$ for each compact $K \subseteq X$.
- μ is *locally finite* if every point $x \in X$ admits a *neighbourhood* (i.e. a set B with $\mathrm{int}(B) \ni x$) of finite measure. Equivalently, each point admits an open neighborhood of finite measure.

Prop. For any topological space X and a Borel measure on it, we have:

(1) μ is finite on compact sets.

X is locally compact

(2) μ is locally finite.

X is 2nd ctbl

(3) μ is σ-finite by open sets.

Proof. (2) $\Rightarrow$ (1). Let $K \subseteq X$ be a compact set. For each $x \in K$, let $U_x \ni x$ be open with finite measure, then $\{U_x\}_{x \in K}$ is an open cover of K, thus $\exists$ finite subcover $U_{x_1}, \dots, U_{x_n}$, so $\mu(K) \leq \sum_{i=1}^{n} \mu(U_{x_i}) < \infty$.

(3) $\Rightarrow$ (2). If $X = \bigcup_n U_n$ with each U_n open and finite measure, then each pt $x \in X$ belongs to one of these U_n.

(1) $\Rightarrow$ (2). Suppose X is locally compact, i.e. every $x \in X$ admits a compact neighbourhood K, i.e. $x \in \mathrm{int}(K)$. But (1) says that $\mu(K) < \infty$, so $\mu(\mathrm{int}(K)) < \infty$, witnessing (2).

(2) $\Rightarrow$ (3). Let $\mathcal{U}$ be a ctbl basis for X. By (2), each pt $x \in X$ has an open neighbourhood $V \ni x$ of finite measure. But this V is union of some sets in $\mathcal{U}$ hence $\exists\, U \in \mathcal{U}$ with $x \in U \subseteq V$, so U also has finite measure. Thus, each x has a $U \in \mathcal{U}$ with $x \in U$ of finite measure. Hence, letting $\mathcal{U}' := \{U \in \mathcal{U} : \mu(U) < \infty\}$, we get that $X = \bigcup_{U \in \mathcal{U}'} U$. □

Def. A Borel measure μ on a metric space X is called **tight** if "μ concentrates on compact sets", i.e. for every measurable set $B \subseteq X$

$$\mu(B) = \sup\{\mu(K) : K \subseteq B \text{ compact}\}.$$

Theorem. Every finite Borel measure μ on a Polish metric space X is tight.